CALCULATING the INVERSE of an INVOLUTE

By Fred P. Eberle

IN MANY VARIATIONS ON A CENTRAL THEME, THE INVOLUTE TOOTH FORM IS USED IN A VAST ARRAY OF PRODUCT DESIGNS. READ ON FOR A DISCUSSION OF TWO USEFUL FUNCTIONS.

The involute tooth form is the only tooth form that provides true conjugate action normal to the tangency of the tooth curves passing through the pitch point. The ramifications of this property give the involute gear family the ability to transmit constant angular acceleration without slippage. In a special case of the straight sided rack: the radius of curvature of the involute is infinite, thus the involute becomes a straight line. Hence, the teeth are straight sided. The involute rack will run with any gear of the same module and pressure angle. And the rack and pinion is particularly useful in transmitting uniform rotational motion into linear motion. Also, changes in center distance—provided contact ratio of the mating mesh is 1.0 or greater—will not alter the rotational velocity of the gear set. This is why the involute tooth form and its variants are so widely used today.

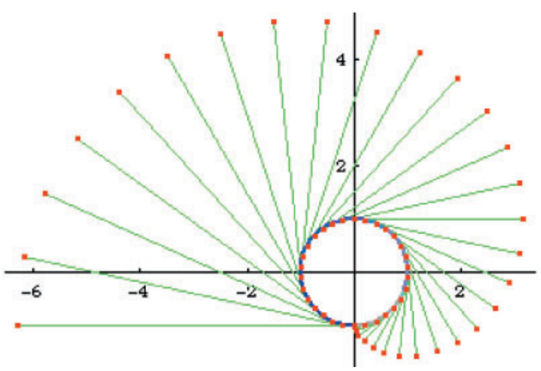


FIGURE 1: The involute curve is determined by the locus of points that are generated by a line unwound on it's base circle. Once the base circle is known the involute can be completely defined.

There are several other tooth forms that have been developed and applied for special purposes, such as:

- Cycloidal
- Hypocycloidal
- Epicycloid
- Trochocentric
- Beveloid
- Spiroid

Each form has its own unique purpose. For this discussion, however, we will limit ourselves to the involute and sevolute functions. The applications of these two entities are indispensable to the vast majority of parallel axis gearing, cams, splines, and serrations in use today.

In a variety of calculations it is very beneficial to determine the inverse of the involute. It is especially helpful in the analysis of tooth thickness and its indirect measurement by means of pins, blocks, or balls.

- Where Involute ϕ = (Tan ϕ - ϕ) where "ϕ" is in radians.

- Sevolute $\varepsilon = [1/\cos(\varepsilon)] - \mathrm{inv}(\varepsilon) = [1/\cos(\varepsilon)] - \mathrm{Tan}(\varepsilon) - (\varepsilon)$

If the standard pressure angle is known (ϕ) at the standard pitch point, then the pressure angle at the base radius is determined by:

(ϕb) = Arcos (db/dp) Where: db = Diameter of Base Circle
dp = Diameter of Pitch Circle

Calculation of the pressure angle any point on the involute curve can then be determined by:

(ϕi) = Arcos(Rb/Ri) Where: Rb = Radius of Base Circle
Ri = Radius where (ϕi) is located

Application of the Involute Function

Some examples where these calculations become very helpful are in the determination of:

- Operating pressure angles
- Gearing on non-standard center distances
- Tight mesh gear rolling (composite inspection) with master gears. More formally: the calculation of master gear "test radius"
- Calculations in profile shifted gears
- Over pin or ball dimensions that do not use standard pin or ball sizes
- Determination of the pointed tooth diameter
- Determination of effective diameter in the cases of gear tip relief or tip radii
- Determination of the profile inspection diameter (start & end), sometimes called the control diameters
- Tooth land thickness in gearing having modified tooth parameters
- Tooth thickness calculations at any point along its involute curve

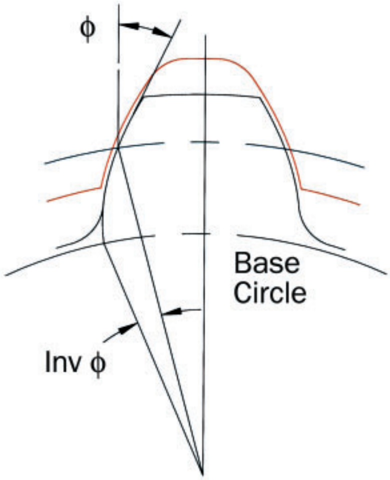
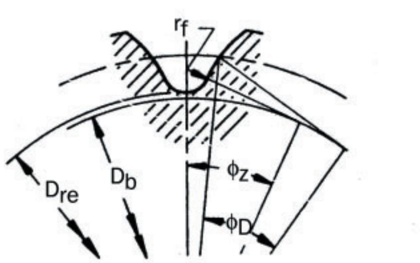


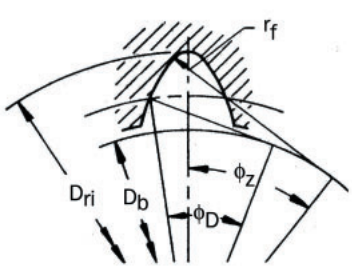
FIGURE 2



$$\text{sevolute } \phi_z = \frac{D_{re}}{D_b} - \left(\frac{t}{D} + \text{inv } \phi_D\right) + \frac{\pi}{N}$$

$$r_f = 0.5(D_b \sec - \phi_z D_{re})$$

FIGURE 3



$$\text{sevolute } \phi_z = \frac{D_{ri}}{D_b} - \left(\frac{s}{D} + \text{inv } \phi_D\right)$$

$$r_f = 0.5(D_{ri} - D_b \sec\phi_z)$$

FIGURE 4

Historical Inverse Involute Calculations

Because there is no direct method for determination of this parameter, several approximations and iterative techniques have been developed (see figure 2). The first approximation is from [Dudley]$_1$:

If inv $\phi < 0.5$, Let
$\phi i = 1.441 (\text{inv } \phi)^{1/3} - 0.366 (\text{inv } \phi)$

If inv $\phi > 0.5$, Let
$\phi i = 0.243\,\pi + 0.471 (\text{inv } \phi)$

These equations can be helpful in determining the first approximation of this parameter before further iteration.

In 1992 [Harry Cheng]$_2$ proposed a derivation of an explicit solution of the inverse involute function where (inv ϕ) is known and the angle (ϕ) is to be found. Using the "asymptotic" series $f(\phi) = \text{inv}^{-1}(\phi)$, the explicit equation becomes:

Where $(\theta) = \text{Tan}(\phi) - \phi$

$$\phi = (3\theta)^{1/3} - (2\theta)/5 + (9/175)\,(3)^{2/3}\,(\theta)^{5/3} - (2/175)\,(3)^{1/3}\,(\theta)^{7/3} - (144/67375)\,(\theta)^3 + (3258/3128125)\,(3)^{2/3}\,(\theta)^{11/3} - (49711/153278125)\,(3)^{1/3}\,(\theta)^{13/3}\ldots\ldots..$$

This approximation, having high precision, is applicable for inv $\phi < 1.8$. Another technique is to use Newton's method of iteration:

For......... $\phi i + 1 = \phi i - f(\phi i)/f(\phi i)$

Rearranging and setting equation to zero yields...... $f(\phi i) = \tan \phi i - \phi i - \text{inv } \phi i = 0$

Solving for inv (ϕi) : the definition of the involute function is : $\text{inv } \phi i = \tan \phi i - \phi i$

Then taking the derivative of the function: $f'(\phi i) = \sec^2 (\phi i) - 1 - 0$

And from trigometric identities we know that: $\sec^2 \theta - 1 = \tan^2 \theta$

We can use substitution to obtain:

$$\phi i + 1 = \phi i + [(\text{inv } \phi i + \phi i - \tan \phi i)/ \tan^2 \phi i]$$

This result led to the development of yet another technique by Irving Laskin. An iteration method was put forth by [Laskin]$_3$ and is suitable for inv ϕ values up to (1.0). Since the corresponding pressure angle is nearly 65 degrees, it is particularly useful for any calculations involving spur and helical involute tooth forms. In his method he shows the first approximation to be:

$\phi_1 = 1.441 (I)^{1/3} - 0.374 (I)$ Where $(I) = \text{inv } \phi = \text{Tan}\phi - \phi$

The second approximation is taken as ϕ_2:

$$\phi_2 = \phi_1 + [\, I - (\text{inv } \phi_1)] / (\text{Tan } \phi_1)^2$$

$$\phi_3 = \phi_2 + [\, I - (\text{inv } \phi_2)] / (\text{Tan } \phi_2)^2 \ldots\ldots...$$

For involute angles up to 30 degrees with two approximations, there is no error to six significant digits. With four approximations this is true to 64.87 degrees. Similarly, for the sevolute function, the first approximation is:

Where $(S) = \varepsilon = [1/\cos(\varepsilon)] - \mathrm{inv}(\varepsilon)$

$= [1/\cos(\varepsilon)] - \mathrm{Tan}(\varepsilon) - (\varepsilon)$

$\varepsilon_1 = 0.8\,(S\text{-}1) + 1.4\,(S\text{-}1)^{1/2}$

$\varepsilon_2 = \varepsilon_1 + [S - (\mathrm{Sev}\ \varepsilon_1)]\ [\ 1 + (1/\sin \varepsilon_1)]$

$\varepsilon_3 = \varepsilon_2 + [S - (\mathrm{Sev}\ \varepsilon_2)]\ [\ 1 + (1/\sin \varepsilon_2)]$...........

For sevolute angles up to 30 degrees with two approximations, there is no error to five significant digits. With three approximations there is no error to eight significant digits and up to 82 degrees of angle.

FIGURE 5: Iteration example - Visual basic code

```
INV_Angle(Inv as Double) As Double

    Dim InvCal As Double
    Dim PI as Double
    Dim Angle_0 As Double
    Dim Angle_1 As Double
    Dim InvDif As Double
    Dim AngleCal As Double

    PI = 3.14
    Angle_0 = 0
    Angle1 = PI / 2
    InvDif = Inv
    AngleCal = 0

    Do while Abs(InvDif) >0.00000001

        AngleCal = (Angle_0 + Angle1) / 2

        InvDif = Tan(AngleCal) - AngleCal - Inv

        If InvDif > 0.0 Then
                Angle_1 = AngleCal
        Else
                Angle_0 = AngleCal
        End If
    Loop

    INV_Angle = AngleCal

End Function
```

Application of the Sevolute Function

The sevolute function has a unique application. It is primarily used in lieu of a generated trochoid to determine the full fillet circular radius that can be used at the gear root diameter. This is particularly useful for powder metal and plastic gears as an

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aid in tool extraction. In addition, a circular radius often adds additional bending strength in design. Another unique use of the sevolute is in providing a full tip radius on splines to aid assembly, or provide sliding bearing contact between the internal and external members.

Since there is no exact solution to solving the inverse functions, the convenient solution is yet another iterative technique, a subroutine, and iterating until convergence is a minimum of six significant digits (see figures 3, 4). This method is convenient to computer programming and is trial and error iteration.

Iteration Example of visual basic code for an exact iterative solution for a minimum of Six significant digits (see figure 5).

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